THE DENSITY OF LONG RUNS OF NON-r-FREE INTEGERS

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ABSTRACT. Fix $r \geq 2$ and let $\Delta_k^{(r)}$ be the natural density of the integers n for which none of $n+1, \dots, n+k$ is r-free. We determine the asymptotics of $\Delta_k^{(r)}$ with an explicit linear term:

$$\log \frac{1}{\Delta_k^{(r)}} = \frac{r-1}{\zeta(r)}\, k \log k + \frac{r}{\zeta(r)}\, k \log\log k - C_r\, k + o(k),$$

$$C_r = \frac{1}{\zeta(r)} \Big((r-1)\log\zeta(r) + r - 1 + r \log \frac{\pi}{r \sin(\pi/r)} \Big).$$

For $r = 2$ this reads $C_2 = \frac{6}{\pi^2}\big(1 + \log\frac{\pi^4}{24}\big) = 1.4595513\ldots$, and the theorem determines Mirsky's constant $\alpha(h)$, the density of the n with $s_{n+1} - s_n = h$ in the sequence of squarefree numbers, to the same precision. The only estimate for $\alpha(h)$ recorded in the literature is $\log \alpha(h) \leq -\frac{5}{4} h \log\log h + O(h)$; ours is smaller by a factor $\asymp \log h / \log\log h$ and is two-sided.

The upper bound rests on the observation that the primes p with $\prod_{p \leq z} p^r \leq \sqrt{k}$ sieve a window of length k with no freedom of alignment whatsoever, so at least $(1/\zeta(r) + o(1))k$ positions must be covered by r-th powers of pairwise distinct primes exceeding $k^{1/r}$; the resulting sum over injections is $m!\, e_m$ with $m \approx k/\zeta(r)$, and e_m is evaluated by a saddle point built on $\sum_p \log(1 + s^r/p^r) = \pi s/(\sin(\pi/r)\log s) + O(s/\log^2 s)$. The matching lower bound comes from the same sieve count together with the Chung–Erdős inequality, which recovers the whole of $m!\, e_m$ and not merely its largest term; this is what makes the linear coefficient explicit.

As a consequence the first-moment threshold for gaps between r-free numbers is $\frac{\zeta(r)}{r-1}\frac{\log x}{\log\log x}\big(1 - \frac{1+o(1)}{r-1}\frac{\log\log\log x}{\log\log x}\big)$. For $r = 2$ the constant is $\pi^2/6$: Erdős's 1951 lower bound for squarefree gaps sits exactly at the first-moment threshold, which is a precise form of his remark that it "seems to be extremely hard to replace $\pi^2/6$ by any larger constant". We emphasise that this does not improve the known range of exponents in the moment problem for squarefree gaps. We show in addition that every congruence class forcing a run of length k has modulus at least $\exp((rq_r + o(1))k\log k)$, so that the Chinese-remainder construction proves only the constant $\zeta(r)/r$ — half of $\zeta(r)/(r-1)$ when $r = 2$; Erdős's lower bound is therefore equivalent to the assertion that the first run of length k occurs at the first-moment threshold rather than at that modulus, and we could not locate a proof of it. All asymptotics are checked against exact values of $\Delta_k^{(2)}$ for $k \leq 150$ and $\Delta_k^{(3)}$ for $k \leq 120$.

1. INTRODUCTION

Let $r \geq 2$. An integer is *r-free* if it is divisible by no r-th power of a prime; the r-free integers have density $1/\zeta(r)$. Write $s_1 < s_2 < \cdots$ for the r-free integers, suppressing r from the notation. For $r = 2$, Erdős [5, (20)] observed that for infinitely many i

$$s_{i+1} - s_i > (1 + o(1))\, \frac{\pi^2}{6}\, \frac{\log s_i}{\log\log s_i}, \tag{1.1}$$

adding that he did not know whether (1.1) had been published before but that it "was certainly known to several mathematicians, e.g. Bateman, Chowla and Mirsky". He continued: "The

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curious thing is that it seems to be extremely hard to replace $\pi^2/6$ by any larger constant, in fact it seems possible that for $i > i_0$

$$s_{i+1} - s_i < (1+\varepsilon)\,\frac{\pi^2}{6}\,\frac{\log s_i}{\log\log s_i}.\text{"} \tag{1.2}$$

Both (1.1) and the conjecture (1.2) appear as Erdős Problem #208 in [1]; (1.2) is open, and even the much weaker assertion $s_{i+1}-s_i \ll_\varepsilon s_i^{\varepsilon}$ is open, the best unconditional exponent being $1/5-\eta$ for a small $\eta>0$ by Pandey [18], after Filaseta and Trifonov [7]. Granville [9] showed that the *abc* conjecture implies $s_{i+1}-s_i \ll_\varepsilon s_i^{\varepsilon}$. For a recent study of the closely related question of which sequences A have infinitely many translates $n+A$ inside the squarefree numbers, see van Doorn and Tao [19].

This paper is about the local quantity underlying (1.1), for every r. Put

$$D_k^{(r)} = \{n \in \mathbb{N} :\ \text{none of } n+1,\dots,n+k \text{ is } r\text{-free}\}, \qquad \Delta_k^{(r)} = \operatorname{dens} D_k^{(r)}. \tag{1.3}$$

The density exists (§2), and $\Delta_k^{(r)} > 0$ for every k by the Chinese remainder theorem. Mirsky [15, Thm. 4] proved that for every $h \ge 1$ the density

$$\alpha_r(h) = \operatorname{dens}\{n :\ n \text{ is } r\text{-free},\ n+1,\dots,n+h-1 \text{ are not},\ n+h \text{ is}\} \tag{1.4}$$

exists; we write $\alpha = \alpha_2$. The two families are related by $\Delta_k^{(r)} = \sum_{h>k}(h-k)\alpha_r(h)$, and §2 shows $\alpha_r(h) = (1+o(1))\Delta_{h-1}^{(r)}$.

Quantitative information about $\alpha(h)$ is what makes the moment problem

$$\sum_{s_{i+1}\le x} (s_{i+1}-s_i)^{\gamma} \sim B(\gamma)\,x, \qquad B(\gamma) = \sum_{h\ge 1} h^{\gamma}\alpha(h), \tag{1.5}$$

tractable; (1.5) is Erdős Problem #145 in [1], proved by Erdős [5] for $\gamma \le 2$, by Hooley [10] for $\gamma \le 3$, and after work of Filaseta [6], Filaseta–Trifonov [8] and Huxley [11, 12] most recently by Chan [2] for $\gamma < 3.75$. The only estimate for $\alpha(h)$ recorded in that literature (see [2, (2)], attributed to [11, Lemma 1]) is

$$\log \alpha(h) \le -\tfrac{5}{4}\,h\log\log h + O(h). \tag{1.6}$$

Our main result determines the true size, with an explicit linear term, for every r.

Theorem 1.1. *Fix $r \ge 2$ and put $q_r = 1/\zeta(r)$ and*

$$C_r = q_r\Big((r-1)\log\zeta(r) + r - 1 + r\log\frac{\pi}{r\sin(\pi/r)}\Big). \tag{1.7}$$

Then, as $k\to\infty$,

$$\log\frac{1}{\Delta_k^{(r)}} = (r-1)q_r\,k\log k + r\,q_r\,k\log\log k - C_r\,k + o(k).$$

The same asymptotic holds for $\log(1/\alpha_r(k+1))$.

For $r=2$ the constant collapses to a pleasant closed form,

$$C_2 = \frac{6}{\pi^2}\Big(1+\log\frac{\pi^4}{24}\Big) = 1.45955133487\ldots, \tag{1.8}$$

and Theorem 1.1 reads $\log(1/\Delta_k^{(2)}) = \frac{6}{\pi^2}k\log k + \frac{12}{\pi^2}k\log\log k - C_2 k + o(k)$. Table 1 lists the constants for small r.

TABLE 1. The three constants of Theorem 1.1.

r	$q_r = 1/\zeta(r)$	$(r-1)q_r$	$r\,q_r$	C_r
2	0.6079271019	0.6079271019	1.215854204	1.45955133487
3	0.8319073726	1.663814745	2.495722118	2.44409748264
4	0.9239384029	2.771815209	3.695753612	3.37918097415
5	0.9643873404	3.857549362	4.821936702	4.31898650447
6	0.9829525923	4.914762961	5.897715554	5.27125777085

What is and is not new. For $r = 2$ the leading term is not new in one direction: the lower bound $\Delta_k^{(2)} \geq \exp(-(1+o(1))q_2 k \log k)$ is, in essence, the construction behind (1.1) and was, as Erdős says, known to Bateman, Chowla and Mirsky. New here are: the matching *upper* bound for $\Delta_k^{(r)}$ with the same constant; the second-order term $r\,q_r\,k \log\log k$; the explicit linear coefficient C_r; and the extension to all $r \geq 2$. Erdős's paper contains no estimate for $\alpha(h)$: his Lemma 1 records only the existence of the density, citing [15], and his Lemma 2 is the different, uniform-in-x statement $\sum_{h>t} \#\{i : s_{i+1} \leq x,\ s_{i+1} - s_i = h\} \ll x/(t^2 \log^2 t)$, which is what his proof of (1.5) for $\gamma \leq 2$ actually uses.

We have not found the analogue of (1.6) for $r \geq 3$ in the literature. A plausible explanation for the absence of any sharp estimate is that none was needed: in (1.5) the constant $\alpha(h)$ enters only through the convergence of $\sum_h h^\gamma \alpha(h)$, and (1.6) already gives that for every γ. We stress the consequence:

Remark 1.2. Theorem 1.1 does *not* enlarge the range of γ in (1.5). The bound (1.6) suffices for the convergence of $B(\gamma)$ for all $\gamma \geq 0$, and the obstruction to $\gamma \geq 3.75$ lies elsewhere entirely — in the lattice point counts discussed in Appendix A, where the gap is a factor 3.2% in a single exponent. The improvement recorded in Corollary 1.3 is a statement about $\alpha(h)$, not about γ.

Two corollaries explain the interest of the upper half of Theorem 1.1.

Corollary 1.3. *For every $\varepsilon > 0$ and all sufficiently large h, $\log \alpha(h) \leq -(q_2 - \varepsilon) h \log h$. This is smaller than (1.6) by a factor $\asymp \log h / \log\log h$, and it is sharp.*

Corollary 1.4. *Let $K_r(x) = \max\{k : \Delta_k^{(r)} \geq 1/x\}$ be the first-moment threshold. Then*

$$K_r(x) = \frac{\zeta(r)}{r-1} \cdot \frac{\log x}{\log\log x} \cdot \Big(1 - \frac{1+o(1)}{r-1} \cdot \frac{\log\log\log x}{\log\log x}\Big).$$

For $r = 2$ the leading constant is $\pi^2/6$. Corollary 1.4 therefore says two things about (1.1) and (1.2). First, $\pi^2/6$ is *exactly* the threshold at which the expected number of runs of length k below x passes from $\gg 1$ to $\ll 1$: no first-moment argument can produce a larger constant. This is a precise form of Erdős's remark. Second, the threshold lies *below* $\zeta(2) \log x / \log\log x$ by a relative $(1+o(1)) \log\log\log x / \log\log x$, so (1.2) is consistent with the first-moment model with room to spare. We stress that this is a statement about the model only; the first-moment method gives no upper bound for an individual gap, and (1.2) remains open.

Method. The proof is elementary and short. Its one point of leverage is Lemma 2.1: if z is chosen so small that $Q = \prod_{p \leq z} p^r \leq \sqrt{k}$, then the number of $i \in [1, k]$ surviving the sieve by the r-th powers p^r, $p \leq z$, is $k \prod_{p \leq z}(1 - p^{-r}) + O(Q)$ *for every* n: these small primes have no freedom of alignment inside a window that already contains many of their periods. Since $\prod_{p \leq z}(1 - p^{-r}) \to q_r$, at least $(q_r - o(1))k$ positions of the window must be covered by r-th

powers of primes exceeding $k^{1/r}$, and such primes are necessarily pairwise distinct because their r-th powers exceed k. Everything reduces to estimating $m!\,e_m$, where e_m is the m-th elementary symmetric function of $\{p^{-r} : p > k^{1/r}\}$ and $m \approx q_r k$.

The novelty in the treatment of $m!\,e_m$, relative to the classical constructions, is that both bounds see the whole of it. The upper bound (§3) is a Chernoff estimate whose saddle point is governed by $\int_0^\infty (1+v^r)^{-1}dv = (\pi/r)/\sin(\pi/r)$. For the lower bound (§4) one cannot simply exhibit disjoint congruence classes attached to the m cheapest primes: that recovers only the largest term of e_m and loses a factor $\exp((r\log\frac{\pi}{r\sin(\pi/r)} + o(1))m)$, which is exactly the part of C_r that carries the arithmetic. Instead we apply the Chung–Erdős inequality to the family of *all* injections into the primes above $k^{1/r}$; the pairwise intersections are controlled by $\sum_{p>k^{1/r}} p^{-r} = o(1/k)\cdot k^{1/r}$, so the loss is only $e^{o(k)}$ and the two bounds meet.

§6 records an exact evaluation of $\Delta_k^{(2)}$ and $\alpha(h)$ for $k, h \le 150$ and of $\Delta_k^{(3)}$ for $k \le 120$, five independent checks on it, a direct numerical confirmation of the coefficient $r\,q_r$ and of the constant C_r, the induced numerical values of $B(\gamma)$, and predictions for the first occurrence of a run of length k for $19 \le k \le 32$ (sequence `A045882` of [17], whose known terms stop at $k = 18$). Appendix A reformulates the exponent in the Huxley–Chan treatment of (1.5).

2. Notation and preliminaries

Throughout, $r \ge 2$ is fixed, p denotes a prime, $\pi(y) = \#\{p \le y\}$, $\theta(y) = \sum_{p\le y}\log p$, P_j is the j-th prime, and $q_r = 1/\zeta(r) = \prod_p(1-p^{-r})$. For a set A of non-negative reals with $\sum_{a\in A} a < \infty$ we write $e_m(A)$ for its m-th elementary symmetric function. Implied constants may depend on r but on nothing else.

Existence of the densities. Fix k. For $y \ge 2$ let $D_k(y)$ be the set of n such that each of $n+1,\dots,n+k$ is divisible by p^r for some $p \le y$. Each $D_k(y)$ is a union of congruence classes modulo $\prod_{p\le y} p^r$, hence has a density $\delta_k(y)$, non-decreasing in y and bounded by 1. Since $0 \le \overline{\mathrm{dens}}\,(D_k^{(r)} \setminus D_k(y)) \le k\sum_{p>y} p^{-r} \to 0$, the density $\Delta_k^{(r)} = \lim_{y\to\infty}\delta_k(y)$ exists. The same argument gives the existence of $\alpha_r(h)$, which is also [15, Thm. 4].

Relation between $\Delta_k^{(r)}$ and $\alpha_r(h)$. A maximal run of exactly ℓ consecutive non-r-free integers contains $\ell - k + 1$ starting points of runs of length k when $\ell \ge k$, and a gap $s_{i+1} - s_i = h$ corresponds to a maximal run of length $h-1$. Hence

$$\Delta_k^{(r)} = \sum_{h>k}(h-k)\,\alpha_r(h), \qquad \text{so} \qquad \alpha_r(k+1) = \Delta_k^{(r)} - 2\Delta_{k+1}^{(r)} + \Delta_{k+2}^{(r)}. \tag{2.1}$$

Since $\alpha_r(h) \le \Delta_{h-1}^{(r)}$, the tail of the first sum is at most $\sum_{j\ge1}(j+1)\Delta_{k+j}^{(r)}$, and Theorem 1.1 — applied to the ratio $\Delta_{k+1}^{(r)}/\Delta_k^{(r)} = \exp(-(r-1)\log k + O(\log\log k))$ — shows this is $O(\Delta_{k+1}^{(r)}) = o(\Delta_k^{(r)})$. Therefore

$$\alpha_r(k+1) = (1+o(1))\,\Delta_k^{(r)}, \tag{2.2}$$

and the two statements of Theorem 1.1 are equivalent. (This is not circular: (2.2) transfers the conclusion and is not used in the proof.)

We shall use four lemmas.

Lemma 2.1 (deterministic sieve count)**.** *Let $z \ge 2$ and $Q = \prod_{p\le z} p^r$. Then for every $n \in \mathbb{Z}$ and every $k \ge 1$,*

$$\#\{1 \le i \le k :\ p^r \nmid n+i \ \text{ for all } p \le z\} \ = \ k\prod_{p\le z}\Big(1-\frac{1}{p^r}\Big) + O(Q).$$

Proof. The condition on i depends only on $i \bmod Q$, and by the Chinese remainder theorem exactly $Q\prod_{p\le z}(1-p^{-r})$ residues modulo Q satisfy it. The interval $[1,k]$ contains $\lfloor k/Q\rfloor$ complete residue systems modulo Q and a remainder of length less than Q. □

Lemma 2.2 (Chernoff bound for e_m)**.** *For any $t>0$, $e_m(A)\le t^{-m}\prod_{a\in A}(1+ta)$.*

Proof. $\prod_{a\in A}(1+ta)=\sum_{j\ge 0}t^j e_j(A)\ge t^m e_m(A)$. □

Lemma 2.3 (Euler product)**.** *As $s\to\infty$,*

$$F_r(s):=\sum_p \log\Big(1+\frac{s^r}{p^r}\Big)=\frac{\pi}{\sin(\pi/r)}\cdot\frac{s}{\log s}+O\Big(\frac{s}{\log^2 s}\Big).$$

Moreover, for $2\le w\le s$, $0\le F_r(s)-\sum_{p>w}\log(1+s^r/p^r)\ll \pi(w)\log s$.

Proof. By partial summation, $F_r(s)=\int_2^\infty \pi(u)\,\frac{r\,s^r}{u(u^r+s^r)}\,du$, the boundary terms vanishing. Inserting $\pi(u)=u/\log u+O(u/\log^2 u)$ and substituting $u=sv$,

$$F_r(s)=r\,s\int_{2/s}^\infty \frac{dv}{(1+v^r)\log(sv)}+O\Big(s\int_{2/s}^\infty\frac{dv}{(1+v^r)\log^2(sv)}\Big).$$

Writing $\log(sv)=\log s\,(1+\log v/\log s)$ and using the classical evaluation $\int_0^\infty(1+v^r)^{-1}dv=(\pi/r)/\sin(\pi/r)$ together with the convergence of $\int_0^\infty|\log v|(1+v^r)^{-1}dv$ yields the stated asymptotic; the range $v<2/s$ contributes $O(1)$ because $\log(sv)\ge\log 2$ there. For the second claim, $0\le\log(1+s^r/p^r)\le r\log s+\log 2$ for $p\ge 2$, so removing the primes $p\le w$ costs at most $\pi(w)(r\log s+\log 2)$. □

Given m large, let $s=s(m)$ be the unique solution of

$$\frac{\pi}{\sin(\pi/r)}\cdot\frac{s}{\log s}=r\,m, \tag{2.3}$$

so that $s=\frac{r\sin(\pi/r)}{\pi}\,m\log s$ and, iterating,

$$\log s=\log m+\log\log m+\log\frac{r\sin(\pi/r)}{\pi}+O\Big(\frac{\log\log m}{\log m}\Big). \tag{2.4}$$

The last lemma is what upgrades Lemma 2.2 to an asymptotic. Recall Darroch's theorem [4]: if X is a sum of independent Bernoulli variables and $\mathbb{E}X$ is an integer, then $\mathbb{E}X$ is a mode of X.

Lemma 2.4 (the Chernoff bound is tight)**.** *Let $A=\{a_1,a_2,\dots\}$ be non-negative with $\sum a_j<\infty$ and let $m\ge 1$ be an integer for which some $t>0$ satisfies $\sum_j\frac{ta_j}{1+ta_j}=m$. Then*

$$t^{-m}\prod_j(1+ta_j)\ \ge\ e_m(A)\ \ge\ \frac{c}{\sqrt{m}}\,t^{-m}\prod_j(1+ta_j)$$

for an absolute constant $c>0$.

Proof. Let $X=\sum_j B_j$ with B_j independent Bernoulli of parameter $\theta_j=ta_j/(1+ta_j)$. Expanding $\prod_j(1-\theta_j+\theta_j)$ shows $\mathbb{P}(X=m)=t^m e_m(A)\prod_j(1+ta_j)^{-1}$, which gives the upper bound (Lemma 2.2) and turns the lower bound into a statement about $\mathbb{P}(X=m)$. By hypothesis $\mathbb{E}X=m$ is an integer, so by Darroch's theorem m is a mode of X. Its variance is $\sigma^2=\sum_j\theta_j(1-\theta_j)\le m$, so by Chebyshev at least $3/4$ of the mass of X lies on the at most $4\sigma+1$ integers within 2σ of m; hence $\mathbb{P}(X=m)=\max_j\mathbb{P}(X=j)\ge\frac34(4\sigma+1)^{-1}\ge c/\sqrt{m}$. □

3. The upper bound

Proposition 3.1. $\log \Delta_k^{(r)} \le -(r-1)q_r\, k\log k - r\, q_r\, k \log\log k + C_r\, k + o(k)$.

Proof. Let k be large and choose $z = z(k)$ maximal with $\theta(z) \le \frac{1}{2r}\log k$; then $z \asymp \log k$ and $Q := \prod_{p\le z} p^r = e^{r\theta(z)} \le \sqrt{k}$.

Step 1: the surviving set. For $n \in \mathbb{Z}$ put $V(n) = \{1 \le i \le k :\ p^r \nmid n+i\ \forall p \le z\}$. By Lemma 2.1 and $Q \le \sqrt{k}$, $|V(n)| = k\, q_{r,z} + O(\sqrt{k})$ with $q_{r,z} = \prod_{p\le z}(1-p^{-r})$, for every n. Since $q_{r,z} = q_r \prod_{p>z}(1-p^{-r})^{-1} = q_r(1+O(z^{1-r}/\log z))$,

$$|V(n)| = q_r\, k\Big(1 + O\Big(\frac{1}{\log^{r-1} k\ \log\log k}\Big)\Big) \qquad \text{for every } n. \tag{3.1}$$

Step 2: removing the middle primes. Let $W(n) = \{i \in V(n) :\ p^r \nmid n+i\ \forall p \le k^{1/r}\}$. The number of $i \in [1,k]$ divisible by p^r for some $p \in (z, k^{1/r}]$ is at most $\sum_{z<p\le k^{1/r}}(k/p^r+1) \ll k/(z^{r-1}\log z) + \pi(k^{1/r}) \ll k/(\log^{r-1} k\, \log\log k)$. Hence, with (3.1), there is a constant C' with

$$|W(n)| \ \ge\ m_1 := \Big\lceil q_r\, k\Big(1 - \frac{C'}{\log^{r-1} k\ \log\log k}\Big)\Big\rceil \qquad \text{for every } n. \tag{3.2}$$

Step 3: an injection into large primes. Suppose $n \in D_k^{(r)}$. Every $i \in W(n)$ has $p^r \mid n+i$ for some prime $p > k^{1/r}$, and for distinct $i, i' \in W(n)$ the corresponding primes are distinct, since $p^r > k > |i-i'|$ forbids $p^r \mid n+i$ and $p^r \mid n+i'$ simultaneously.

Put $Q' = \prod_{p\le k^{1/r}} p^r$ and note that $V(n), W(n)$ depend only on $\rho = n \bmod Q'$; write $W(\rho)$ accordingly and fix a subset $W'(\rho) \subseteq W(\rho)$ of size m_1. Let $\mathcal{P} = \{p : p > k^{1/r}\}$ and let $\Phi(\rho)$ be the set of injections $\varphi : W'(\rho) \to \mathcal{P}$. For $x \ge 1$,

$$\#(D_k^{(r)} \cap [1,x]) \ \le \sum_{\rho \bmod Q'} \ \sum_{\varphi\in\Phi(\rho)} \#\{n \le x :\ n \equiv \rho\ (Q'),\ \varphi(i)^r \mid n+i\ \forall i \in W'(\rho)\}.$$

Although $\Phi(\rho)$ is infinite, for fixed x only finitely many φ contribute: the inner set is empty unless $\prod_i \varphi(i)^r \le x+k$, and there are at most $m_1!$ times the number of r-free integers $\le (x+k)^{1/r}$ such φ, so $O_k(x^{1/r})$ of them. Each nonempty inner set is a single congruence class modulo $Q'\prod_i \varphi(i)^r$, of cardinality at most $x\big(Q'\prod_i \varphi(i)^r\big)^{-1} + 1$. The "+1" terms therefore contribute $O_k(x^{1/r}) = o(x)$ as $x \to \infty$ with k fixed, the sums are absolutely convergent, and

$$\Delta_k^{(r)} \ \le \sum_{\varphi} \prod_{i\in W'} \varphi(i)^{-r} \ =\ m_1!\, e_{m_1}, \qquad e_m := e_m\big(\{p^{-r} : p > k^{1/r}\}\big), \tag{3.3}$$

the sum over injections from a set of size m_1 into $\mathcal{P}$ being independent of ρ, and $\sum_\rho 1/Q' = 1$.

Step 4: estimating $m!\, e_m$. Write $m = m_1$ and let $s = s(m)$ be as in (2.3). By Lemma 2.2 with $t = s^r$ and then Lemma 2.3 with $w = k^{1/r}$,

$$\log e_m \ \le \sum_{p>k^{1/r}} \log\Big(1 + \frac{s^r}{p^r}\Big) - rm\log s \ \le\ \frac{\pi}{\sin(\pi/r)}\frac{s}{\log s} + O\Big(\frac{s}{\log^2 s}\Big) + O\big(k^{1/r}\big) - rm\log s.$$

By (2.3) the first term equals rm, and $s \asymp m\log m \asymp k\log k$ gives $s/\log^2 s \ll m/\log m$. With (2.4),

$$\log e_m \ \le\ rm - rm\log m - rm\log\log m - rm\log\frac{r\sin(\pi/r)}{\pi} + o(m).$$

Since $\log m! = m\log m - m + O(\log m)$,

$$\log(m!\, e_m) \ \le\ -(r-1)m\log m - rm\log\log m + \Big(r - 1 + r\log\frac{\pi}{r\sin(\pi/r)}\Big)m + o(m). \tag{3.4}$$

Finally $m = m_1 = q_r k(1+O(\log^{1-r} k/\log\log k))$, and since the right side of (3.4) has derivative $-(r-1)\log m + O(\log\log m)$ in m, replacing m_1 by $q_r k$ costs only $O(k/\log\log k) = o(k)$. Using $\log(q_r k) = \log k - \log\zeta(r)$ and $\log\log(q_r k) = \log\log k + O(1/\log k)$ turns (3.4) into

$$\log \Delta_k^{(r)} \le -(r-1)q_r k\log k - r q_r k\log\log k + q_r\Big((r-1)\log\zeta(r) + r - 1 + r\log\frac{\pi}{r\sin(\pi/r)}\Big)k + o(k),$$

which is the assertion. □

4. The lower bound

We use the Chung–Erdős inequality [3]: for events $A_1,\dots,A_M$ in a probability space with $\sum_i \mathbb{P}(A_i) > 0$,

$$\mathbb{P}\Big(\bigcup_i A_i\Big) \;\ge\; \frac{\big(\sum_i \mathbb{P}(A_i)\big)^2}{\sum_{i,j}\mathbb{P}(A_i\cap A_j)}. \tag{4.1}$$

Proposition 4.1. $\log \Delta_k^{(r)} \ge -(r-1)q_r\, k\log k - r\, q_r\, k\log\log k + C_r\, k - o(k)$.

Proof. Let z, Q be as in §3, so $Q \le \sqrt{k}$ and $\log Q \le \frac12\log k$. Fix a residue class ρ_0 modulo Q; by Lemma 2.1 the set $V = V(n)$ is the same for all $n \equiv \rho_0\ (Q)$, and by (3.1)

$$m := |V| = q_r\, k\Big(1 + O\Big(\frac{1}{\log^{r-1} k\ \log\log k}\Big)\Big). \tag{4.2}$$

Let $Y > k$ be a parameter, let $\mathcal{P}_Y = \{p : k^{1/r} < p \le Y\}$, and work in the probability space $\prod_{p\in\mathcal{P}_Y}\mathbb{Z}/p^r\mathbb{Z}$ with the uniform measure, which by the Chinese remainder theorem computes densities of sets defined by congruences to the moduli p^r, $p\in\mathcal{P}_Y$, independently of the class ρ_0. For an injection $\varphi : V \to \mathcal{P}_Y$ set

$$A_\varphi = \{n \equiv \rho_0\ (Q) :\ \varphi(i)^r \mid n+i \text{ for all } i\in V\}.$$

If $n\in A_\varphi$ then every $i\in[1,k]\setminus V$ has $n+i$ divisible by p^r for some $p\le z$ and every $i\in V$ has $n+i$ divisible by $\varphi(i)^r$; hence $\bigcup_\varphi A_\varphi \subseteq D_k^{(r)}$ and

$$\Delta_k^{(r)} \;\ge\; \frac{1}{Q}\,\mathbb{P}\Big(\bigcup_\varphi A_\varphi\Big). \tag{4.3}$$

The two sums in (4.1). By the Chinese remainder theorem $\mathbb{P}(A_\varphi) = \prod_{i\in V}\varphi(i)^{-r}$, so

$$\sum_\varphi \mathbb{P}(A_\varphi) = m!\, e_m^{(Y)}, \qquad e_m^{(Y)} := e_m(\{p^{-r} : p\in\mathcal{P}_Y\}). \tag{4.4}$$

For the second sum, let φ,ψ be injections and $D = \{i\in V : \varphi(i)\ne\psi(i)\}$. If some prime is required to divide two of the $n+i$ with distinct i, then $A_\varphi\cap A_\psi = \emptyset$, because $p^r > k > |i-i'|$; otherwise the constraints involve the $m+|D|$ distinct primes $\varphi(V)\cup\psi(D)$ and $\mathbb{P}(A_\varphi\cap A_\psi) = \mathbb{P}(A_\varphi)\prod_{i\in D}\psi(i)^{-r}$. Summing over ψ by first choosing D and then the values of ψ on D,

$$\sum_\psi \mathbb{P}(A_\varphi\cap A_\psi) \;\le\; \mathbb{P}(A_\varphi)\sum_{d\ge0}\binom{m}{d}\sigma^d = \mathbb{P}(A_\varphi)(1+\sigma)^m, \qquad \sigma := \sum_{p>k^{1/r}} p^{-r}. \tag{4.5}$$

By the prime number theorem $\sigma \ll k^{(1-r)/r}/\log k$, so $m\sigma \ll k^{1/r}/\log k = o(k)$ and $(1+\sigma)^m = e^{o(k)}$. Combining (4.1), (4.4) and (4.5),

$$\mathbb{P}\Big(\bigcup_\varphi A_\varphi\Big) \;\ge\; \frac{m!\, e_m^{(Y)}}{(1+\sigma)^m} \;=\; m!\, e_m^{(Y)}\, e^{-o(k)}.$$

Letting $Y \to \infty$ gives $e_m^{(Y)} \uparrow e_m$, whence by (4.3)

$$\log \Delta_k^{(r)} \;\geq\; \log(m!\, e_m) - \log Q - o(k) \;=\; \log(m!\, e_m) - o(k). \tag{4.6}$$

Evaluating $m!\, e_m$ from below. Let $t > 0$ solve $\sum_{p > k^{1/r}} \frac{tp^{-r}}{1+tp^{-r}} = m$; such a t exists because the left side is continuous, increasing, and unbounded. By Lemma 2.4, $\log e_m \geq \sum_{p>k^{1/r}} \log(1 + tp^{-r}) - m \log t - O(\log m)$. Since the right side is the Chernoff bound of §3 at its own minimiser, and since that minimiser is $t = s(m)^r + o(\cdot)$ by (2.3), Lemma 2.3 and (2.4) give

$$\log e_m \;\geq\; rm - rm \log m - rm \log\log m - rm \log \frac{r \sin(\pi/r)}{\pi} - o(m),$$

so that $\log(m!\, e_m)$ equals the right side of (3.4) up to $o(m)$. Feeding this into (4.6) and using (4.2) exactly as at the end of §3 completes the proof. □

Propositions 3.1 and 4.1 prove Theorem 1.1; the statement for α_r follows from (2.2).

Proof of Corollary 1.3. Immediate from Theorem 1.1 with $r = 2$ and (2.2), since $k \log k$ dominates $k \log\log k$ and k. □

Proof of Corollary 1.4. Write $L = \log(1/\Delta_k^{(r)}) = (r-1)q_r k(\log k + \frac{r}{r-1} \log\log k + O(1))$ and $\lambda = \log L$. Then $\lambda = \log k + \log\log k + \log((r-1)q_r) + O(\log\log k/\log k)$, so $\log k = \lambda - \log\log k - \log((r-1)q_r) + O(\log\log k/\log k)$ and hence

$$L = (r-1)q_r\, k\Big(\lambda + \tfrac{1}{r-1}\log\log k + O(1)\Big), \qquad k = \frac{\zeta(r)}{r-1} \cdot \frac{L}{\lambda}\Big(1 + \frac{\frac{1}{r-1}\log\log k + O(1)}{\lambda}\Big)^{-1}.$$

Taking $x = e^L$ we have $\lambda = \log\log x$ and $\log\log k = \log\lambda + o(1) = \log\log\log x + o(1)$, which is the assertion. The passage from the equality $\Delta_k^{(r)} = 1/x$ to the definition of $K_r(x)$ costs $O(1)$ in k, which is absorbed in the $o(1)$ because $\Delta_{k+1}^{(r)}/\Delta_k^{(r)} = \exp(-(r-1)\log k + O(\log\log k))$. □

5. The constructive threshold, and what (1.1) asserts

Theorem 1.1 identifies $\zeta(r)/(r-1)$ as the first-moment constant. It is worth contrasting this with what the classical Chinese-remainder construction delivers, because the two differ by a factor $r/(r-1)$ — a factor 2 when $r = 2$.

Definition. A *covering system of length k* is a finite set P of primes together with residues $\rho_p \bmod p^r$, $p \in P$, such that for every $i \in [1,k]$ there is a $p \in P$ with $\rho_p \equiv -i \pmod{p^r}$. Its *modulus* is $M = \prod_{p \in P} p^r$.

Any n with $n \equiv \rho_p \ (p^r)$ for all $p \in P$ lies in $D_k^{(r)}$, and by the Chinese remainder theorem the least such n is smaller than M. This is exactly the device behind (1.1), and behind our own lower bound in §4.

Proposition 5.1 (the constructive threshold)**.** *Every covering system of length k has modulus*

$$M \;\geq\; \exp\big((r\, q_r + o(1))\, k \log k\big).$$

Consequently a covering system exhibits a run of length k only below $\exp((rq_r + o(1))k\log k)$, and the construction proves no more than: for infinitely many i,

$$s_{i+1} - s_i > (1 + o(1))\, \frac{\zeta(r)}{r} \cdot \frac{\log s_i}{\log\log s_i},$$

which for $r = 2$ is the constant $\pi^2/12$. The bound on M is attained, by §4.

Proof. Let z be maximal with $\prod_{p\le z} p^r \le \sqrt{k}$, so $z \asymp \log k$, and let n be a solution of the system. By Lemma 2.1, at least $q_r k(1+o(1))$ of the indices $i \le k$ satisfy $p^r \nmid n+i$ for all $p \le z$; by the computation in Step 2 of §3, all but $O(k/(\log^{r-1} k \log\log k))$ of those additionally satisfy $p^r \nmid n+i$ for all $p \le k^{1/r}$. Each such i is covered by some $p \in P$ with $p > k^{1/r}$, and distinct such i are covered by distinct primes, since $p^r > k > |i-i'|$ prevents $p^r \mid n+i$ and $p^r \mid n+i'$ simultaneously. Hence P contains at least $m = q_r k(1+o(1))$ primes exceeding $k^{1/r}$, and

$$\log M \;\ge\; r\big(\theta(P_{\pi(k^{1/r})+m}) - \theta(k^{1/r})\big) = r\,m\log m\,(1+o(1)) = r\,q_r\,k\log k\,(1+o(1))$$

by the prime number theorem, exactly as in §4. For the second statement put $\log x = \log M$ and invert as in Corollary 1.4; the exponent is larger by the factor $r/(r-1)$, so the resulting constant is smaller by that factor. □

Remark 5.2. Comparing Proposition 5.1 with Corollary 1.4, for $r=2$ the two thresholds are

$$\underbrace{\frac{\pi^2}{12}\cdot\frac{\log x}{\log\log x}}_{\text{covering systems}} \qquad\text{and}\qquad \underbrace{\frac{\pi^2}{6}\cdot\frac{\log x}{\log\log x}}_{\text{first moment}}.$$

They differ because $\Delta_k^{(r)}$ exceeds the density $1/M$ of a single covering system by the factor $m!\,e_m\prod_{j\le m} p_j^r = \exp((q_r+o(1))k\log k)$ coming from the $m!$ assignments of primes to positions — precisely the gain that §4 was designed to capture, and the reason the Chung–Erdős step there cannot be replaced by the exhibition of one class. Erdős states (1.1) with the constant $\pi^2/6$ without a detailed proof, remarking only that it "was certainly known to several mathematicians, e.g. Bateman, Chowla and Mirsky". In view of Proposition 5.1, (1.1) is *equivalent* to the assertion that the first run of length k occurs near the first-moment threshold $\Delta_k^{-1+o(1)}$ and not merely near the covering-system modulus $\Delta_k^{-2+o(1)}$; no covering system can establish it. We have not been able to locate an argument of the required kind in the literature.

Question 5.3. Is there a proof of (1.1) with the constant $\pi^2/6$? Equivalently, is the first occurrence $F(k)$ of a run of k consecutive non-squarefree integers bounded by $\Delta_k^{-1+o(1)}$?

The numerical evidence for Question 5.3 is strong: by Table 4 and Figure 3 the ratio $F(k)\Delta_k$ lies in $[0.17, 3.67]$ for all eighteen known records, that is $F(k) = \Delta_k^{-1+o(1)}$ over seventeen orders of magnitude, whereas the covering-system modulus is already $\Delta_k^{-2+o(1)}$. Whatever the status of (1.1), Corollary 1.4 shows that $\pi^2/6$ cannot be improved by any first-moment argument, which is the content of Erdős's remark that it "seems to be extremely hard to replace $\pi^2/6$ by any larger constant".

6. Exact values, checks, and numerical confirmation

6.1. **An exact evaluation.** The following is the standard Mirsky-type inclusion–exclusion; what is new is only the scale at which we run it. Let Z be the set of primes with $p^r \le k$ and $M = \prod_{p\in Z} p^r$; for $r=2$ and $k \le 168$ one may take $Z = \{2,3,5,7,11\}$, $M = 5\,336\,100$, and for $r=3$ and $k \le 124$ one may take $Z = \{2,3\}$, $M = 216$. Every prime $p \notin Z$ then has $p^r > k$ and so covers at most one position of a window of length k, with probability p^{-r}, independently of $n \bmod M$. Consequently, if $u(\rho)$ denotes the number of positions of $[1,k]$ not covered by p^r, $p \in Z$, when $n \equiv \rho\ (M)$, then

$$\Delta_k^{(r)} = \frac{1}{M}\sum_{\rho \bmod M} Q(u(\rho)), \qquad Q(u) = \sum_{j=0}^{u}(-1)^j\binom{u}{j}G(j), \qquad G(j) = \prod_{p\notin Z}\Big(1-\frac{j}{p^r}\Big), \quad (6.1)$$

and this is exact. The histogram of $u(\rho)$ is computed by one pass of a sliding window over $\mathbb{Z}/M\mathbb{Z}$; the products $G(j)$ are evaluated from prime zeta values, and the alternating sum defining $Q(u)$ is carried out in 900-digit arithmetic (the cancellation is severe: for $r = 2$, $k = 150$ the terms have size $\approx 10^{40}$ and the result is $\approx 10^{-239}$). The same method with $G(j)$ replaced by $G(j+2)$ evaluates $\alpha_r(h)$, since requiring the two endpoints to be uncovered as well simply adds two positions to the inclusion–exclusion.

6.2. **Five checks.** (i) For $r = 2$, $\Delta_1^{(2)}$ agrees with $1 - 6/\pi^2 = 0.39207289814597337134\ldots$ to 20 digits; for $r = 3$, $\Delta_1^{(3)}$ agrees with $1 - 1/\zeta(3) = 0.168092627419\ldots$ to 14 digits. (ii) $\sum_h \alpha(h) = 6/\pi^2$ and $\sum_h h\,\alpha(h) = 1$ hold to 20 digits, the latter being the statement that the gaps tile $\mathbb{N}$. (iii) $\alpha(1)/\sum_h \alpha(h) = 0.53071182\ldots$ and $\alpha(2)/\sum_h \alpha(h) = 0.324294\ldots$ agree with the constants recorded in [17, A076259]. (iv) A segmented sieve over $[1, 10^{12}]$, using a wheel modulo $2^2 3^2 5^2 7^2$ and marking only the squares of primes ≥ 11, gives the counts of Table 2; the longest run below 10^{12} is 13, consistent with $F(14) = 1.0435 \cdot 10^{12}$. (v) The identity $\alpha(k+1) = \Delta_k - 2\Delta_{k+1} + \Delta_{k+2}$ of (2.1) holds to the printed precision for all $k \leq 148$.

TABLE 2. Sieve check for $r = 2$. $N_k = \#(D_k^{(2)} \cap [1, 10^{12}])$ against $\Delta_k^{(2)} \cdot 10^{12}$.

k	N_k (sieved)	$\Delta_k^{(2)} \cdot 10^{12}$	ratio
1	392 072 897 726	392 072 898 146	1.0000000
3	18 634 011 558	18 634 010 350	1.0000001
5	641 333 605	641 332 384	1.0000019
7	5 206 950	5 206 270	1.000131
9	73 177	73 077	1.00137
11	725	754	0.961
13	8	15.5	(Poisson)

6.3. **The constant C_r.** Theorem 1.1 asserts that

$$R_r(k) := \frac{1}{k}\Big(\log \frac{1}{\Delta_k^{(r)}} - (r-1)q_r\, k \log k - r\, q_r\, k \log\log k\Big) \longrightarrow -C_r.$$

Figure 1 plots $R_r(k) + C_r$ for $r = 2$, $10 \leq k \leq 150$ and for $r = 3$, $10 \leq k \leq 120$. Both curves descend towards 0, from 0.96 and 0.94 respectively to 0.11 and 0.09; note that no parameter has been fitted — $C_2 = 1.45955\ldots$ and $C_3 = 2.44410\ldots$ come from the closed form (1.7). The residual is consistent with the $O(m \log\log m/\log m)$ error carried by (2.4), which at $k = 150$ amounts to about 0.39 per unit of k.

6.4. **Confirmation of the coefficient $r\,q_r$.** The $\log\log$ term cannot be seen by fitting Δ_k directly for $k \leq 150$; see Remark 6.1. It can be confirmed by isolating the factor $Q(u)$ of (6.1), which is the pure "coupon collector" quantity and carries the whole of the second-order term. For $r = 2$ the saddle point analysis of §3 predicts, with $s = s(u)$ defined by (2.3),

$$\frac{d}{du}\log\frac{1}{Q(u)} = \log u + 2\log\log s(u) + 2\log\tfrac{2}{\pi} + o(1), \qquad 2\log\tfrac{2}{\pi} = -0.9032\ldots \tag{6.2}$$

Figure 2 compares (6.2) with the exact symmetric difference quotient of $\log(1/Q(u))$ computed from the 900-digit values of Q. The gap decreases monotonically from 0.374 at $u = 30$ to 0.022 at $u = 145$, i.e. by a factor 17, which leaves no room for a coefficient other than 2.

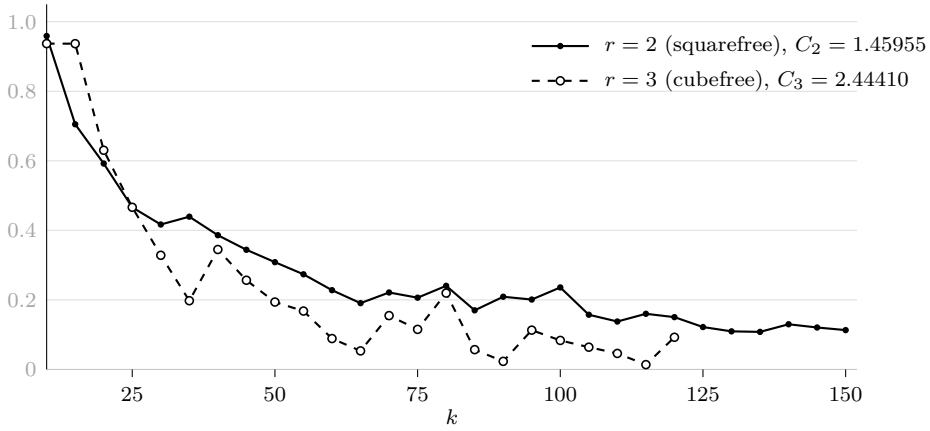

FIGURE 1. $R_r(k) + C_r$ for $r = 2$ and $r = 3$, with C_r given by the closed form (1.7). Both curves tend to 0, confirming the linear coefficient of Theorem 1.1. The oscillation is arithmetic, not noise; compare Figure 3.

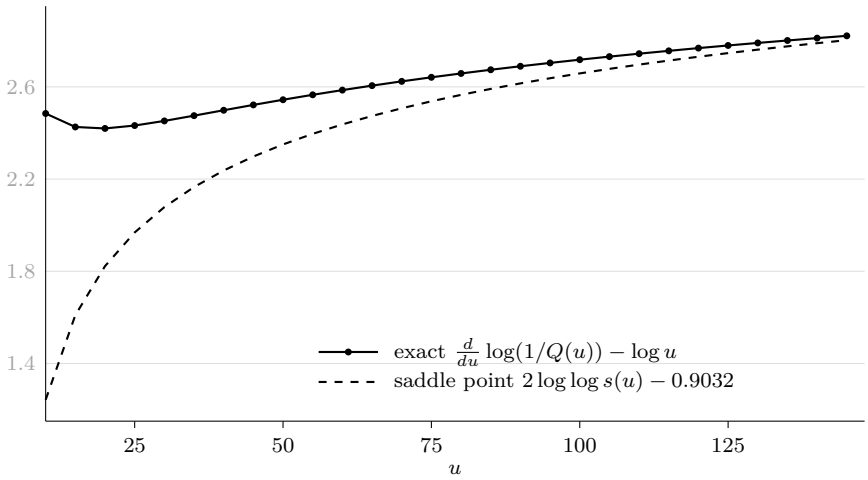

FIGURE 2. Confirmation of the coefficient $r\,q_r$ for $r = 2$. The two curves approach one another; their difference falls from 0.374 at $u = 30$ to 0.022 at $u = 145$.

Solving the saddle equation with the actual primes (exactly for $p \le 2 \cdot 10^7$, by the prime number theorem integral beyond) allows (6.2) to be extrapolated far beyond the range in which $Q(u)$ itself is computable. Table 3 shows the resulting ratio creeping towards 2 at the predicted rate $2 - 0.589/\log\log u$; at $u = 10^{30}$ it is still only 1.81, which is exactly why a direct fit for $k \le 150$ is hopeless.

6.5. **Values for $r = 2$, and the quality of $1/\Delta_k$ as a predictor.** Table 4 lists $\Delta_k^{(2)}$ and $\alpha(k+1)$ for $k \le 20$ together with the first occurrence $F(k)$ of a run of length k (A045882) and the ratio $F(k)\Delta_k$. Over the whole known range the ratio has geometric mean 0.908 and lies in $[0.17, 3.67]$: across seventeen orders of magnitude $1/\Delta_k$ predicts $F(k)$ to within a small constant factor. Figure 3 displays this together with the predictions of §6.7.

TABLE 3. $(\frac{d}{du}\log(1/Q(u)) - \log u)/\log\log u$ for $r = 2$, from the saddle point with the actual primes. The limit is 2.

u	10^2	10^4	10^6	10^9	10^{12}	10^{18}	10^{30}
ratio	1.776	1.749	1.767	1.780	1.788	1.799	1.812
$2 - 0.589/\log\log u$	1.614	1.735	1.776	1.806	1.823	1.842	1.861

TABLE 4. Exact densities for $r = 2$, and the first occurrence of a run of length k.

k	$\Delta_k^{(2)}$	$\alpha(k+1)$	$F(k)$	$F(k)\Delta_k$
1	0.392073	0.197147	4	1.568
2	0.10678	0.0716601	8	0.854
3	0.018634	0.0149788	48	0.894
4	0.00214826	0.000938535	242	0.520
5	0.000641332	0.000500664	844	0.541
6	$7.29375 \cdot 10^{-5}$	$6.27919 \cdot 10^{-5}$	22020	1.606
7	$5.20627 \cdot 10^{-6}$	$4.74547 \cdot 10^{-6}$	217070	1.130
8	$2.66941 \cdot 10^{-7}$	$1.25699 \cdot 10^{-7}$	1092747	0.292
9	$7.30768 \cdot 10^{-8}$	$6.40077 \cdot 10^{-8}$	8870024	0.648
10	$4.91166 \cdot 10^{-9}$	$3.44829 \cdot 10^{-9}$	221167422	1.086
11	$7.54184 \cdot 10^{-10}$	$6.79634 \cdot 10^{-10}$	221167422	0.167
12	$4.50046 \cdot 10^{-11}$	$1.46822 \cdot 10^{-11}$	47255689915	2.127
13	$1.54595 \cdot 10^{-11}$	$1.42864 \cdot 10^{-11}$	82462576220	1.275
14	$5.96503 \cdot 10^{-13}$	$5.57143 \cdot 10^{-13}$	1043460553364	0.622
15	$1.99409 \cdot 10^{-14}$	$1.90509 \cdot 10^{-14}$	79180770078548	1.579
16	$5.22193 \cdot 10^{-16}$	$2.17484 \cdot 10^{-16}$	3215226335143218	1.679
17	$1.54404 \cdot 10^{-16}$	$1.46306 \cdot 10^{-16}$	23742453640900972	3.666
18	$4.09765 \cdot 10^{-18}$	$3.91514 \cdot 10^{-18}$	125781000834058568	0.515
19	$9.79825 \cdot 10^{-20}$	$7.75381 \cdot 10^{-20}$	—	—
20	$1.34579 \cdot 10^{-20}$	$6.56585 \cdot 10^{-22}$	—	—

6.6. **The constants $B(\gamma)$.** The constants in (1.5) are now numerically accessible. Since $\alpha(h)$ decays like $\exp(-q_2 h \log h)$ by Theorem 1.1, the series $B(\gamma) = \sum_h h^\gamma \alpha(h)$ converges for every $\gamma \geq 0$ and truncating at $h = 60$ is more than sufficient. We find

$$B(0) = \tfrac{6}{\pi^2}, \quad B(1) = 1, \quad B(2) = 2.04070977646714\ldots, \quad B(3) = 5.04286811382262\ldots,$$
$$B(3.6875) = 10.2852261048655\ldots, \qquad B(3.75) = 11.0089932906174\ldots,$$
$$B(4) = 14.5232122094753\ldots$$

($B(0)$ and $B(1)$ are forced and serve as checks; $B(2)$, $B(3)$ and $B(3.75)$ are the constants in the theorems of Erdős, Hooley and Chan respectively).

6.7. **Predictions.** Marmet [14, Appendix A] already gave estimates of this kind, based on an approximation to the probability of assembling the minimum number of primes needed to produce a gap of a given length, and displayed them against the known values. We replace his approximate probability by the exact density Δ_k and add a calibration factor. Assuming that runs of a given length occur at the scale predicted by their density — a hypothesis validated by Table 4 and Figure 3 over seventeen orders of magnitude — the first occurrence $F(k)$ should

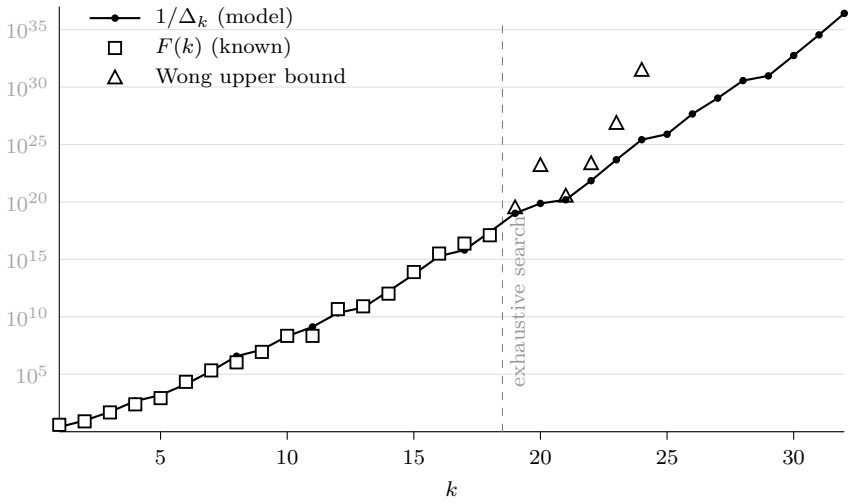

FIGURE 3. The model $1/\Delta_k^{(2)}$ against the known first occurrences $F(k)$ and against E. B. Wong's Chinese-remainder upper bounds for $F(k)$, $19 \le k \le 24$, recorded in [14]. The vertical axis is $\log_{10}$. The model passes through all eighteen known records and stays below all six upper bounds. The visible zigzag is arithmetic, not noise: it reflects the residue structure modulo 4 and 9 (for instance $\Delta_{20}/\Delta_{21} = 2.1$ while $\Delta_{21}/\Delta_{22} = 46$).

be $\approx 0.908/\Delta_k$. Table 5 records this for $19 \le k \le 32$, with an uncertainty of roughly a factor 4 either way. Exhaustive search reached $1.2587 \cdot 10^{17}$ in [14] and 10^{18} in [16]. All the predictions lie below the Chinese-remainder upper bounds of E. B. Wong recorded in [14], and for $k = 21$ the prediction is only a factor 2.3 below Wong's bound.

TABLE 5. Predicted first occurrence of a run of k consecutive non-squarefree integers, and Wong's upper bounds where available.

k	$1/\Delta_k$	prediction $0.908/\Delta_k$	Wong upper bound
19	$1.021 \cdot 10^{19}$	$9.27 \cdot 10^{18}$	$3.13 \cdot 10^{19}$
20	$7.431 \cdot 10^{19}$	$6.75 \cdot 10^{19}$	$1.48 \cdot 10^{23}$
21	$1.545 \cdot 10^{20}$	$1.40 \cdot 10^{20}$	$3.21 \cdot 10^{20}$
22	$7.063 \cdot 10^{21}$	$6.41 \cdot 10^{21}$	$2.14 \cdot 10^{23}$
23	$4.695 \cdot 10^{23}$	$4.26 \cdot 10^{23}$	$6.87 \cdot 10^{26}$
24	$2.627 \cdot 10^{25}$	$2.39 \cdot 10^{25}$	$2.85 \cdot 10^{31}$
25	$7.814 \cdot 10^{25}$	$7.09 \cdot 10^{25}$	—
26	$4.540 \cdot 10^{27}$	$4.12 \cdot 10^{27}$	—
27	$1.085 \cdot 10^{29}$	$9.85 \cdot 10^{28}$	—
28	$3.647 \cdot 10^{30}$	$3.31 \cdot 10^{30}$	—
29	$9.399 \cdot 10^{30}$	$8.53 \cdot 10^{30}$	—
30	$5.563 \cdot 10^{32}$	$5.05 \cdot 10^{32}$	—
31	$3.480 \cdot 10^{34}$	$3.16 \cdot 10^{34}$	—
32	$2.589 \cdot 10^{36}$	$2.35 \cdot 10^{36}$	—

Remark 6.1 (why the $\log\log$ term is numerically invisible)**.** A naive fit of $\log(1/\Delta_k^{(2)}) = q_2 k(\log k + A\log\log k + C)$ to the exact values for $k \le 150$ returns $A \approx 0.8$, not 2. There are two reasons. First, $\log\log k$ ranges only over $[1.2, 1.6]$ for $30 \le k \le 150$, so the fit has no leverage and any A can be compensated by C. Second, in that range the sum (6.1) is dominated not by the typical value of $u(\rho)$ but by its *minimum*: for $k = 150$ the mean of $u(\rho)$ is 93.26 while the single term $u = 89 = \min_\rho u(\rho)$ carries 98.7% of Δ_{150}. This is a finite-k alignment effect, possible only because the window length is far below the period M; it disappears as k grows and is precisely what Lemma 2.1 rules out in the asymptotic regime. §6.4 confirms the coefficient nonetheless, and §6.3 confirms C_r directly.

Appendix A. A reformulation of the exponent in the moment problem

This appendix is not used elsewhere in the paper, and it is not a rigorous derivation: it is a bookkeeping exercise on the published arguments of Huxley [11] and Chan [2], recorded because it locates the obstruction referred to in Remark 1.2 in a single inequality. We have not verified it against [11], to which we did not have access; the two numerical coincidences below are our only evidence that the bookkeeping is faithful.

In that treatment one splits the gap length H and the prime scale P dyadically and must show, for each pair (H, P), either that $T(H,P) < H/(64\gamma\log H)$, where $T(H,P)$ is the largest number of primes $p \in [P, 2P)$ whose square has a multiple in some window of length H, or one of three bounds for a sextuple count $S(H,P)$. Applying the r-th derivative test of Huxley–Sargos [13] to $f_n(u) = n/u^2$ with $M = P$, $\delta = H/P^2$, $\lambda_r \asymp x/P^{r+2}$, its first two terms $M\lambda_r^{2/(r(r+1))}$ and $M\delta^{2/((r-1)(r-2))}$ give the two thresholds $P \le H^a x^{-b}$ and $P \le H^c$ with

$$a = \frac{r(r+1)}{r^2-r-4}, \qquad b = \frac{2}{r^2-r-4}, \qquad c = \frac{1-e}{1-2e}, \quad e = \frac{2}{(r-1)(r-2)}.$$

Matching these against the requirement $P \gg H^{(2\gamma-4)/3}$ of Case 2 and against the overlap condition $H \le x^{3/(8\gamma-13)}$ yields the two caps

$$\gamma_1(r) = \frac{3a+13b+4}{8b+2} = \frac{7r^2-r+10}{2(r^2-r+4)}, \qquad \gamma_2(r) = 2 + \tfrac{3}{2}c. \tag{A.1}$$

Table 6 evaluates these. That $\gamma_1(4) = 59/16$ and $\gamma_1(5) = 15/4$ agree, with no free parameters, with the exponents of [12] and [2] is the evidence referred to above. The table also indicates why $r = 5$ is the best choice: γ_1 is largest at $r = 6$, but there γ_2 has already fallen back to $59/16$.

Table 6. The two caps (A.1) on the moment exponent as a function of the order r of the derivative test.

r	$\gamma_1(r)$	$\gamma_2(r)$	min	
4	$59/16 = 3.6875$	5	3.6875	cf. [12]
5	$15/4 = 3.7500$	$31/8 = 3.8750$	**3.7500**	cf. [2]
6	$64/17 = 3.7647$	$59/16 = 3.6875$	3.6875	capped by γ_2
7	$173/46 = 3.7609$	$47/13 = 3.6154$	3.6154	
8	$15/4 = 3.7500$	$68/19 = 3.5789$	3.5789	
10	$175/47 = 3.7234$	$241/68 = 3.5441$	3.5441	

A general threshold $P \le H^a x^{-b}$ would reach the ceiling $\gamma < 19/5$ imposed by Case 1 (whose bound $H^{6/5}x^{3/5}$ is critical at $H = x^{1/5}$) precisely when $a \ge 5.8\,b + 1.2$; in the derivative-test

parametrisation $a = 1/(1-(r+2)\kappa)$, $b = \kappa a$ this reads $\kappa \geq 0.2/(1.2r - 3.4)$, i.e. $\kappa \geq 1/13$ at $r = 5$ against the available $\kappa = 2/(r(r+1)) = 1/15$. Improving γ beyond 3.75 at all would require only $\kappa \geq 0.0688$, a gain of 3.2% in a single exponent, at the single critical configuration $H = x^{0.17564}$, $P = x^{0.20609}$. We have not been able to supply it.

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The computations were carried out in Python (`mpmath`) and C; the code and the tables of $\Delta_k^{(r)}$ and $\alpha_r(h)$ are available from the author.

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